LESSON

SETS

Introduction

The concept of set serves as a fundamental part of the present day mathematics. Today this concept is being used in almost every branch of mathematics. Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc, required the knowledge of sets

Sets and their Representations

Set: Set is a well-defined collection of objects.

The following points may be noted with respect to sets

(i) Objects, elements and members of a set are synonymous terms.
(ii) Sets are usually denoted by capital letters A, B, C, X, Y, Z, etc.
(iii) The elements of a set are represented by small letters a, b, c, x, y, z, etc.

If a is an element of a set A, we say that "a belongs to A" the Greek symbol $\in$(epsilon) is used to denote the phrase 'belongs to'. Thus, we write a $\in$ A. If 'b' is not an element of a set A, we write b $\notin$ A and read "b does not belong to A".

Thus, in the set V of vowels in the English alphabet, a $\in$ V but b $\notin$ V. In the set P of prime factors of 30, $3 \in P$ but 15 $\notin$ P.

There are two methods of representing a set.

1. Roster or tabular form
2. Set-builder form

(1) In roster form, all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }.

Example:

(i) The set of all even positive integers less than 7 is described in roster form as {2,4,6}.
(ii) The set of all vowels in the English alphabet is {a, e, i, o, u}.
(iii) The set of odd natural numbers is represented by {1,3,5,...........}. The dots tell us that the list of odd numbers continue indefinitely.

Note:

(i) In roster form, the order in which the elements are listed is immaterial. The set of all natural numbers which divide 42 can be represented as {1,2,3,6,7,14,21,42} as well as {1,3,7,21,2,6,14,42}.
(ii) It maybe noted that while writing the set in roster form an element is not generally repeated, i.e., all the elements are taken as distinct. For example, the set of letters forming the word 'SCHOOL' is {S, C, H, O, L} or {H, O, L, C, S}. Here, the order of listing elements has no relevance.

(2) In set-builder form, all the elements of a set possess a single common property which is not possessed by any element outside the set. For example, in the set {a, e, i, o, u}, all the elements possess a common property, namely, each of them is a vowel in the English alphabet, and no other letter possess this property. Denoting this set by V, we write
V = {x : x is a vowel in English alphabet}

It may be observed that we describe the element of the set by using a symbol x (any other symbol like the letters y, z, etc. could be used) which is followed by a colon ":". After the sign of colon, we write the characteristic property possessed by the elements of the set and then enclose the whole description within braces. The above description of the set V is read as "the set of all x such that x is a vowel of the English alphabet". In this description the braces stand for "the set of all", the colon stands for "such that".

Examples:

A = {x : x is a natural number and $3 < x < 10$}

B = {x : x is a natural number which divides 42}

C = {y : y is a vowel in the English alphabet}

D = {z : z is an odd natural number}

The Empty Set

Definition: A set which does not contain any element is called the empty set or the null set or the void set.

The empty set is denoted by the symbol ϕ or { }.

Examples:

(i) Let A = {x : $1 < x < 2$, x is a natural number}. Then A is the empty set, because there is no natural number between 1 and 2.

(ii) B = {x : $x^2 - 2 = 0$ and x is rational number}. Then B is the empty set because the equation $x^2 - 2 = 0$ is not satisfied by any rational value of x.

(iii) C = {x : x is an even prime number greater than 2}. Then C is the empty set, because 2 is the only even prime number.

(iv) D = {x : $x^2 = 4$, x is odd}. Then D is the empty set, because the equation $x^2 = 4$ is not satisfied by any odd value of x.

Finite and Infinite Sets

Definition: A set which is empty or consists of a definite number of elements is called finite otherwise, the set is called infinite.

Consider some examples:

(i) Let W be the set of the days of the week. Then W is finite.

(ii) Let S be the set of solutions of the equation $x^2 - 6 = 0$. Then S is finite.

(iii) Let G be the set of points on a line. Then G is infinite.

When we represent a set in the roster form, we write all the elements of the set within braces { }. It is not possible to write all the elements of an infinite set within braces { } because the numbers of elements of such a set is not finite. So, we represent some infinite set in the roster form by writing a few elements which clearly indicate the structure of the set followed (or preceded) by three dots.

For example, {1,2,3.......} is the set of natural numbers, {1,3,5,7....} is the set of odd natural numbers, {..., –3, –2, –1, 0, 1, 3,......} is the set of integers. All these sets are infinite.

Note: All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do to follow any particular pattern.

Equal Sets

Given two sets A and B, if every element of A is also an element of B and if every element of B is also an element of A, then the sets A and B are said to be equal.

Definition: Two sets A and B are said to be equal if they have exactly the same elements and we write $A = B$. Otherwise, the sets are said to be unequal and we write $A \neq B$.

We consider the following examples:
(i) Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 1, 4, 2\}$. Then $A = B$.
(ii) Let A be the set of prime numbers less than 6 and P the set of prime factors of 30. Then A and P are equal, since 2, 3 and 5 are the only prime factors of 30 and also these are less than 6.

Note: A set does not change if one or more elements of the set are repeated. For example, the sets $A = \{1,2,3\}$ and $B = \{2,2,1,3,3\}$ are equal, since each element of A is in B and vice-versa..

Subsets

Definition: A set A is said to be a subset of a set B if every element of A is also an element of B.
In other words,

$$A \subset B \text{ if } a \in A \Rightarrow a \in B$$

We read the above statement as "A is a subset of B if a is an element of A implies that a is also an element of B". If A is not a subset of B, we write $A \not\subset B$.

We may note that for A to be a subset of B, all that is needed is that every element of A is in B. It is possible that every element of B may or may not be in A. If every element of B is also in A, then we shall also have $B \subset A$. In this case, A and B are the same sets so that we have $A \subset B$ and $B \subset A$ $\Leftrightarrow$ $A = B$.

Note: (i) Every set A is a subset of itself, i.e., $A \subset A$.
(ii) ϕ is a subset of every set.
(iii) $N \subset Z \subset Q, Q \subset R, T \subset R, N \not\subset T$

Examples:

(i) The set Q of rational numbers is a subset of the set R of real numbers, and we write $Q \subset R$.
(ii) If A is the set of all divisors of 56 and B the set of all prime divisors of 56, then B is a subset of A and we write $B \subset A$.
(iii) Let $A = \{1,3,5\}$ and $B = \{x : x \text{ is an odd natural number less than } 6\}$. Then $A \subset B$ and $B \subset A$ and hence $A = B$.
(iv) Let $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d\}$. Then A is not a subset of B, also B is not a subset of A.

Proper subset and superset

Let A and B two sets. If $A \subset B$ and $A \neq B$, then A is called a proper subset of B and B is called superset of A. For example,
$A = \{1,2,3\}$ is a proper subset of $B = \{1,2,3,4\}$.

Singleton Set

If a set A has only one element, we call it is singleton set. Thus, $\{a\}$ is a singleton set.

Intervals as subsets of R: Let a, b $\in$ R and a < b. Then the set of real numbers {y : a < y < b] is called an open interval and is denoted by (a, b). All the points between a and b belong to the open interval (a, b) but a, b themselves do not belong to this interval.

The interval which contains the end points also is called closed interval and is denoted by [a, b]. Thus

$$[a, b] = \{x : a \le x \le b\}$$

We can also have intervals closed at one end and open at the other, i.e.,

$[a, b) = \{x : a \le x < b\}$ is an open interval from a to b, including a but excluding b.

$(a, b] = \{x : a < x \le b\}$ is an open interval from a to b including b but excluding a.

Note: The number (b–a) is called the length of any of the intervals (a, b), [a, b], [a, b) or (a, b].

Power Set

Definition: The collection of all subsets of a set A is called the power set of A. It is denoted by P(A). In P(A) every element is a set

Thus, if A = {1, 2}, then

$$P(A) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$$

Also, note that n [P(A)] = $4 = 2^2$

In general, if A is a set with n(A) = m, then n [P(A)] = 2^m.

Universal Set

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called the universal set and is denoted by U.

Venn Diagrams

Most of the relationships between sets can be represented by means of diagrams which are known as Venn diagrams.These diagrams consists of rectangles, and closed curves usually circles. The universal set is represented usually by a rectangle and its subsets by circles.

In Venn diagrams, the elements of the sets are written in their respective circles.

Illustration 1 in Fig U = {1,2,3,...10} is the universal set of which

A = {2,4,6,8,10} is a subset.

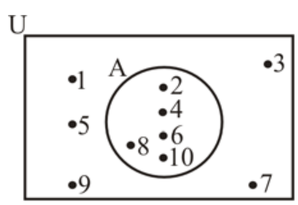

Fig.

Illustration 2: In Fig. , U = {1, 2, 3,.....10} is the universal set of which

A = {2, 4, 6, 8, 10} and B = {4, 6} are subsets, and also B $\subset$ A.

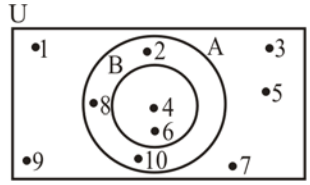

Fig.

Operations on Sets

1. Union of sets:

Definition: The union of two sets A and B is the set which consists of all the elements of A and all the elements of B, the common elements being taken only once. The symbol '$\cup$' is used to denote union. Symbolically, we write $A \cup B$ and usually read as 'A union B'..

The union of two sets can be represented by a Venn diagram as shown in Fig.

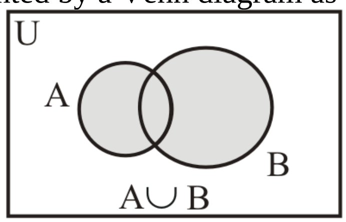

Fig.

The shaded portion in Fig. represents $A \cup B$

Some Properties of the Operation of Union

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $(A \cup B) \cup C = A \cup (B \cup C)$
(Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of U)

Intersection of sets

Definition: The intersection of two sets A and B is the set of all those elements which belong to both A and B. The symbol '$\cap$' is used to denote the intersection. Symbolically, we write
$A \cap B = \{x: x \in A \text{ and } x \in B\}$

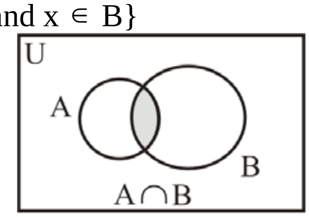

Fig.

The shaded portion in Fig. indicates the intersection of A and B.

Disjoint Sets:

If A and B are two sets such that $A \cap B = \phi$, then A and B are called disjoint sets.

For example, let A = {2,4,6,8} and

B = {1,3,5,7}. Then A and B are disjoint sets, because there are no elements which are common to A and B. The disjoint sets can be represented by means of Venn diagram as shown in the Fig. In the above diagram, A and B are disjoint sets.

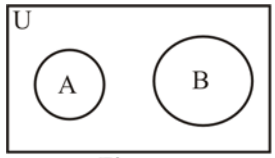

Fig.

Some Properties of the Operation of Intersection.

(vi) $A \cap B = B \cap A$ (Commutative law)

(vii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

(iii) $\phi \cap A = \phi$, $U \cap A = A$ (Law of ϕ and U)

(iv) $A \cap A = A$ (Idempotent law)

(viii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law) i.e., $\cap$ distributes over $\cup$

Difference of Sets: The difference of the sets A and B in this order is the set of elements which belong to A but not to B. Symbolically, we write A – B and read as "A minus B".

i.e. $A - B = \{x : x \in A \text{ and } x \notin B\}$. The differences of A and B can be represented by venn diagram as shown in Fig.

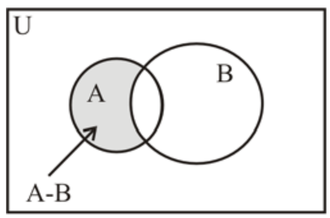

Fig.

Complement of a Set

Definition: Let U be the universal set and A a subset of U. Then the complement of A is the set of all elements of U which are not the elements of A. Symbolically, we write A' to denote the complement of A with respect to U. Thus,

$A' = \{x : x \in U \text{ and } x \notin A\}$. Obviously $A' = U - A$

The complement A' of a set A can be represented by a Venn diagram as shown in Fig.

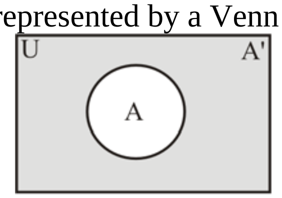

Fig.

The shaded portion represents the complement of the set A.

Some Properties of Complements Sets

1. Complement laws : (i) $A \cup A' = U$ (ii) $A \cap A' = \phi$
2. De Morgan's law : (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$
3. Law of double complementation : $(A')' = A$
4. Laws of empty set and universal set $\phi' = U$ and $U' = \phi$

Note: Let A and B be finite sets. If $A \cap B = \phi$, then

(i) $n(A \cup B) = n(A) + n(B)$ ....(1)

In general, if A and B are finite Sets, then.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(ii) If A, B, and C are finite Sets, then.

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

If A is a subset of the universal set U, then its complement A is also a subset of U.